Effects of noise and confidence thresholds in nominal and metric Axelrod dynamics of social influence

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We study the effects of bounded confidence thresholds and of interaction and external noise on Axelrod's model of social influence. Our study is based on a combination of numerical simulations and an integration of the mean-field master equation describing the system in the thermodynamic limit. We find that interaction thresholds affect the system only quantitatively, but that they do not alter the basic phase structure. The known crossover between an ordered and a disordered state in finite systems subject to external noise persists in models with general confidence threshold. Interaction noise here facilitates the dynamics and reduces relaxation times. We also study Axelrod systems with metric features and point out similarities and differences compared to models with nominal features.

I. INTRODUCTION

Given the increasing influence of mass media, globalization, electronic communication, and intercontinental travel, the apparent persistence of cultural diversity seems surprising. To study this problem Axelrod [1] proposed a simple agent-based model to study how cultural features disseminate. In particular, the model addresses the question of how cultural diversity can result from locally attractive interaction. Axelrod's model is—in the language of physics and dynamical systems theory—a cellular automaton with a set of discrete degrees of freedom placed on a discrete spatial lattice, updated in time through specific interaction rules, and is easily simulated computationally. The simulation and theoretical analysis of social and economic systems have in the recent years been taken up by the statistical physics community (see, e.g., [2–6] and references therein). Social systems are here interpreted as many-particle problems and tools originally developed to study physical systems have been transferred and adapted to the study of socio-economic models in different contexts. See also [7] for a sociological perspective of agent-based modeling.

One of the first systematic studies of the Axelrod model is the one of [8], where a phase transition between a monocultural state (referred to as globalization) and a multicultural phase (referred to as polarization) has been identified as a function of the degree of variation of the initial conditions chosen before running the Axelrod automaton. In terms of physics this transition is one between ordered and disordered states. Axelrod's two-body interaction is attractive as such and drives the system to order. Two particles can only interact, however, if their states are not fully distinct from each

other. These interaction barriers lead to potential jamming. The model thus exhibits some similarity with kinetically constrained models of glass-forming materials [9,10]. If the disorder of the initial conditions is higher than some critical value, the kinetic arrest occurs after a finite number of time steps, and the system remains jammed in a disordered state. For low enough disorder in the random initial conditions, however, the ordering dynamics continues indefinitely in infinite systems. In finite systems full convergence is reached at finite times, and the dynamics comes to a halt when any remaining disorder has been eliminated, resulting in a fully monocultural "globalized" state.

A variety of extensions and variations of Axelrod's original model have been proposed. Klemm *et al.* [11,12], for example, studied the effects of noise on the jamming behavior of Axelrod systems; moreover, the effects of mass media influence have recently been addressed in [13,14]. The model, originally defined on a square lattice, has furthermore been simulated on a variety of complex networks in order to study the effects of the topology of the underlying web of interactions on the dynamics and convergence properties [15].

One of the obvious shortcomings of Axelrod's original model and the subsequent variations is the fact that a metric structure is missing in opinion space. Opinion σ_{if} of agent i on feature f takes values in a discrete set, usually labeled by $\{1, \ldots, q\}$, and agents are classified only as to whether they have the same opinion on a certain issue ($\sigma_{if} = \sigma_{jf}$), or whether their opinions are different ($\sigma_{if} \neq \sigma_{jf}$). No notion of partial agreement on a feature is present. Agents can either fully agree on a certain issue f or they disagree (a more formal definition of the model will follow below). This assumption has been relaxed in a sociological context, for example, in [16–18], and the notion of metric spaces has been introduced. For such features a gradual distinction of a degree of agreement can be defined, for example, given by

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$|\sigma_{if} - \sigma_{jf}|^m$, where the integer m is referred to as "moderation" [16]. A study of a multidimensional opinion model with continuous states has been presented in [19].

As a second drawback, no notion of different tolerance levels or inclination to change one's opinion is present in the original setup. Interaction between agents in Axelrod's original model can occur once they agree on at least one out of the total number F of features. The mentioned quantities F and q are the main model parameters in Axelrod's original setup. From a sociological point of view it is interesting to study the model in more generality and to introduce a "confidence threshold" $\vartheta \in \{0, \ldots, F\}$, so that agents have the potential to interact only if they agree on (strictly) more than ϑ out of F issues. This is referred to as "bounded confidence" in the sociological literature. Axelrod's original model corresponds to minimal confidence threshold ($\vartheta = 0$), in which interaction is possible rather easily. Large values of the threshold ϑ systematically suppress the potential to interact, corresponding to more and more conservative agents, who do not change opinion easily.

Studies of different models addressing either of these two points e.g., through the introduction of continuous opinion states can be found in [19–26]. A first analysis of the effects of metric features and confidence thresholds in the context of the Axelrod model has been conducted in [17,18]. These studies focus mostly on numerical simulations and are mostly restricted to specific choices of the model parameters F and q. The aim of the current work is to complement and extend the analysis of [17,18] through a more general study of the model in parameter space. We also provide analytical results based on a master equation approach [8] for the Axelrod model with nominal features and general confidence threshold.

II. MODEL

The system is composed of $N = L \times L$ agents fixed on the nodes of square lattice of lateral extension L. For simplicity we consider periodic boundary conditions in both spatial dimensions. The state of agent $i \in \{1, \ldots, N\}$ is characterized by an opinion vector $\vec{\sigma}_i = (\sigma_{i1}, \ldots, \sigma_{iF})$, where the integer $F > 1$ denotes the number of cultural "features" in the model. Each component σ_{if} then indicates the opinion of agent i on issue f. In Axelrod's original formulation each component σ_{if} takes one of the q values $\{1, \ldots, q\}$ at each time step, so that each spin $\vec{\sigma}_i$ describes one of q^F cultures. Initially each σ_{if} is drawn at random from the set $\{1, \ldots, q\}$ with no correlations between agents or features. The model parameter q hence measures the degree of disorder in the random initial spin configuration.

We will in the following distinguish between "nominal" and "metric" features as suggested in [17,18]. We first describe the dynamics of the nominal Axelrod model. Here, the system evolves in time by iteration of the following steps:

(i) Select one spin $i \in \{1, \ldots, N\}$ at random. Subsequently select one of its four nearest neighbors at random. Call this second spin j.

(ii) Compute the overlap $\omega(i,j) = \Sigma_{f=1}^{F} \delta_{\sigma_{if}, \sigma_{jf}} \in \{0, \ldots, F\}$ between spins i and j (with $\delta_{\sigma, \sigma'}$ the Kronecker delta).

(iii) If $\omega(i,j) = F$ continue with (v) (spins $\vec{\sigma}_i$ and $\vec{\sigma}_j$ are in identical states).

(iv) Set the probability for i and j to interact to $I = \delta$ if $\omega(i,j) \leq \vartheta$ and to $I = \omega(i,j)/F$ if $\omega(i,j) > \vartheta$. Then with probability $1 - I$ leave spins i and j unchanged. With probability I spins i and j perform the following interaction: choose one feature f at random such that $\sigma_{if} \neq \sigma_{jf}$. Such a feature exists as $\omega(i,j) < F$. Then set $\sigma_{if} = \sigma_{jf}$.

(v) External noise. With probability r perform the following: choose one spin i and one feature f at random. Set σ_{if} to a value chosen randomly from $\{1, \ldots, q\}$.

(vi) Resume at (i).

We will refer to one cycle (i)–(vi) as a *microscopic* time step in the following. At system size N the duration of such a step is taken to be $\Delta t = 1/N$. In general we will present the time evolution of the system mostly in terms of *macroscopic* time units t, so that one unit of time t corresponds to N microscopic interaction cycles, i.e., on average to one (attempted) update per spin.

In the above dynamics, ϑ is the interaction threshold mentioned above. In the absence of interaction noise ($\delta = 0$) neighboring agents have the potential to interact if and only if they share opinions on (strictly) more than ϑ out of F features. To soften this constraint we follow [17] and introduce a source of noise, allowing agents who agree on ϑ or fewer features to interact with probability δ. We refer to this type of stochasticity as "interaction noise" in the following; δ measures its strength. r in the above update rules instead denotes the strength of what we will call "external noise." After each time step, with probability r a randomly chosen component of a randomly chosen spin is set to a random value $\{1, \ldots, q\}$. This type of noise has first been studied in [11].

III. MASTER EQUATION IN THE MEAN-FIELD APPROXIMATION

In this section we will consider a mean-field approximation of the model. In our mean-field model it is possible and convenient to consider the dynamics in terms of bonds, i.e., of pairs of neighboring agents, rather than in terms of spins $\{\vec{\sigma}_i\}$. Following the strategy of [8,27,28], let $P_m(t)$ be the probability that, at a given time t, a bond is of type m, i.e., that the two agents at the ends of the bond have the same opinion on exactly m features. We will occasionally refer to bonds of type F as "fully saturated" in the following. If we let ρ be the probability that at the starting point $t = 0$ of the dynamics two spins share a given feature, we have

$$P_m(0) = \binom{F}{m} \rho^m (1 - \rho)^{F-m}. \tag{1}$$

For σ_{if} drawn independently and with equal probabilities from $\{1, \ldots, q\}$ one has $\rho = 1/q$.

We further define λ to be the probability that two independent spin components are equal to one another, but different from a given third. The probability λ is in principle a time-dependent quantity as the system evolves according the Axelrod dynamics. We here neglect this time dependence and assume that λ is well approximated by its initial value
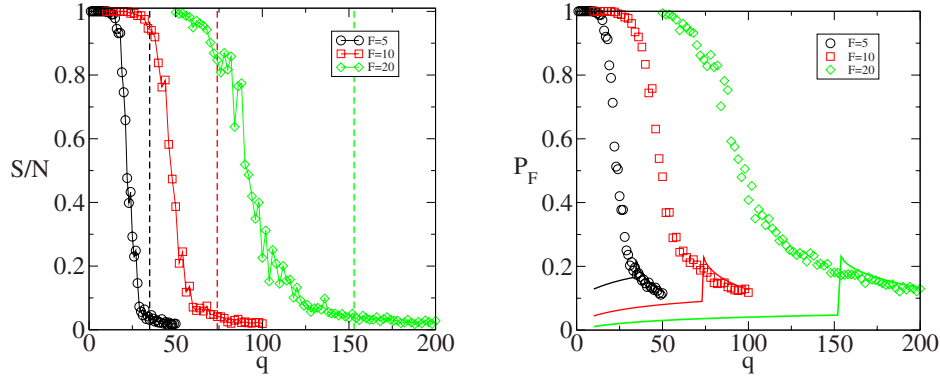
FIG. 1. (Color online) Relative size S/N of the largest cultural region in the standard Axelrod model with F nominal features; initial conditions are drawn from $\sigma_{if} \in \{1, \ldots, q\}$ with flat distribution. Symbols show data from simulation run until convergence, system size $N = 400$, averaged over ten samples, and $F = 5, 10, 20$. The vertical dashed lines mark the location of the phase transition as predicted by a numerical integration of the master equation. The right panel shows P_F from simulations (symbols) compared to results predicted by the master equation (solid lines).

$\lambda = (q-1)^{-1}$ throughout the dynamics. This was seen not to have any significant effects on results in [8,27]. Now denote by $W_{n,m}^{(k)}$ the transition probability that a bond of type n becomes of type m due to the updating of a neighboring bond of type k. The only nonzero elements are [8,27,28]

$$W_{n,n-1}^{(k)} = n/F,$$

$$W_{n,n}^{(k)} = (1 - \lambda)(1 - n/F),$$

$$W_{n,n+1}^{(k)} = \lambda(1 - n/F),$$

independent of k. We will therefore suppress the superscript k in the following.

Let us further define I_k to be the probability with which two agents who share opinions on k features interact if selected for potential update. One has

$$I_k = \begin{cases} \delta, & k \leq \vartheta, \\ k/F, & \vartheta + 1 \leq k < F, \\ 0, & k = F. \end{cases}$$

The master equation can then be written in the form

$$\frac{g}{2} \frac{dP_m(t)}{dt} = \sum_{k=0}^{F-1} [\delta_{m,k+1} - \delta_{m,k}] I_k P_k + (g-1)$$

$$\times \left(\sum_{k=0}^{F-1} I_k P_k \right) \sum_{n=0}^{F} (P_n W_{n,m} - P_m W_{m,n})$$

$$+ rg \left[(1 - \delta_{m,F}) P_{m+1} \frac{m+1}{F} \left(1 - \frac{1}{q} \right) \right.$$

$$- (1 - \delta_{m,F}) P_m \frac{m}{F} \frac{1}{q}$$

$$+ (1 - \delta_{m,0}) P_{m-1} \left(1 - \frac{m-1}{F} \right) \frac{1}{q}$$

$$\left. - (1 - \delta_{m,0}) P_m \frac{m}{F} \left(1 - \frac{1}{q} \right) \right]. \quad (2)$$

This equation is an approximation in the mean-field sense, and the thermodynamic limit is implied. The master equation can be expected to describe the system at most at large system sizes and will therefore not be able to capture features that are characteristic of finite systems. The geometry of the square lattice is mimicked, in the mean-field spirit, by the prefactors $g-1$ and g in the different terms of the master equation. g here denotes the coordination number of each spin, so that these coefficients reflect the number of spins with whom a given spin can interact (albeit these are not nearest neighbors any longer). On a square lattice in two dimensions one has $g = 4$. The prefactor $g/2$ in front of the time derivative in Eq. (2) takes into account the fact that the system contains $g/2$ bonds per lattice site. One would expect the master equation to be accurate in the case of degree-regular graphs (of connectivity g), as discussed for example in [28]. Still, as demonstrated [8] and as we will see below in the context of external and interaction noise, an approach based on numerical integration of the master equation is able to reproduce some features of the two-dimensional model at least qualitatively.

IV. AXELROD DYNAMICS WITH NOMINAL FEATURES

A. Base line model

For completeness we reproduce the behavior of the base line Axelrod model ($\theta = r = \delta = 0$) in Fig. 1. For any given number $F > 2$ of features a discontinuous transition between an ordered state at $q < q_c(F)$ and a disordered phase at larger values of q is observed. At $q < q_c(F)$ the coarsening dynamics of the model persists until a fully ordered state is reached. For any feature $f = 1, \ldots, F$ all agents then agree on one opinion, i.e., $\sigma_{if} = \sigma_{jf}$ for all i and j. In finite systems such a state is reached after a finite time. Figure 1 depicts the relative size S/N of the largest culturally homogeneous region of spins as a function of q. A homogeneous region R is here defined as a subset of the $L \times L$ agents, so that within R all agents agree on all features [29]. As seen in the figure, one finds only one region at convergence for $q < q_c(F)$ and has $S/N = 1$. At val-
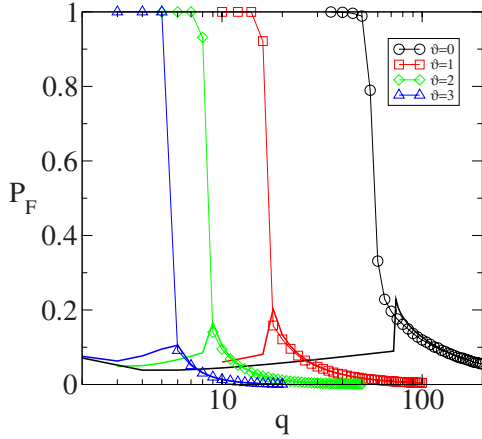
FIG. 2. (Color online) Fraction of fully saturated bonds P_F in the Axelrod model with $F=10$ nominal features and with confidence threshold $\vartheta=3,2,1,0$ (from left to right); initial conditions are drawn from $\sigma_{if} \in \{1,\dots,q\}$ with flat distribution. Connected markers are from simulations with $N=100^2$ agents, run until convergence, and averaged over ten samples. Thick solid lines show theoretical predictions by the master equation.
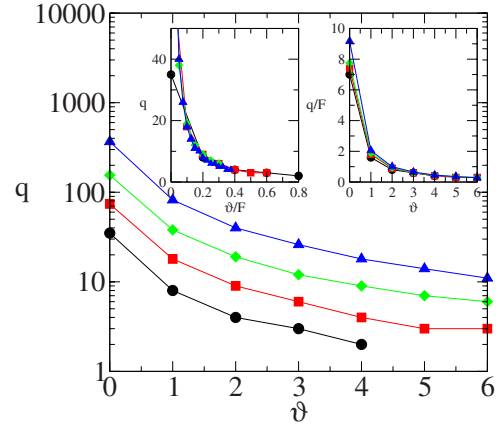

FIG. 3. (Color online) Phase diagram of the model with general confidence threshold in the mean-field approximation. Locations of the transitions are obtained from the master equation. The curves show $F=40,20,10,5$ from top to bottom. The system is in the disordered phase above the respective lines. Ordered states can be expected below. The insets show a rescaling in terms of the relative threshold ϑ/F (left) and of the relative number of opinion states per feature q/F (right).

ues of q larger than $q_c(F)$ dynamic arrest occurs before the system can reach a fully ordered phase. After the arrest no further ordering is possible due to the kinetic constraints imposed on the otherwise attractive spin dynamics. The system remains in a disordered state, marked by a large number of small cultural regions and a vanishing number of active bonds. S/N remains small at convergence in this regime. As seen in Fig. 1 the fraction P_F of fully saturated bonds behaves similarly to S/N at convergence and can be well captured by the master equation in the disordered regime. Since P_F offers a measure of order which can, in the disordered phase, be obtained from the master equation and from simulations allowing for a quantitative comparison, we will mostly use P_F rather than S/N in the following as an order parameter. The ordering at low values of q cannot be obtained from an approach based on the master equation [8]. We also note that the precise form of the curves P_F versus q (as shown, e.g., in Figs. 1 and 2) may depend on the specific details of the model definitions. In [8], for example, spin states are not strictly limited to the set $\{1,\dots,q\}$ and qualitative differences can hence occur. Integrating the master equation at different choices of F furthermore indicates that the shape of such curves may depend on the number of features (see, e.g., [28] for the case $F=2$).

B. Effects of interaction threshold

We now turn to the noise-free model with general interaction threshold ϑ. As shown in Fig. 2 the qualitative behavior of the model is not affected much by the introduction of a confidence threshold. As before, an ordered phase is found at low values of q and a disordered one at q larger than some critical value $q_c(\vartheta,F)$. One finds that an increased confidence threshold suppresses interaction and hence reduces the range of q in which order can be reached. At fixed F, $q_c(\vartheta,F)$ is a decreasing function of ϑ. Figure 2 also demon-

strates that the master equation given above describes the qualitative behavior of the system and dependence on the interaction threshold appropriately. In the disordered phase even a reasonable quantitative agreement between numerical measurements of the fraction of fully saturated bonds and the corresponding theoretical predictions can be observed. We attribute remaining discrepancies to the mean-field approximation, inaccuracies in capturing the two-dimensional geometry, and to finite-size effects.

Figure 3 depicts the phase diagram of the model in the (ϑ,q) plane for different values of F, as obtained from the master equation [30]. The disordered phase is found at large values of ϑ and q, respectively, while order is reached at low q and/or ϑ. It may here be interesting to ask whether the relevant variable is the absolute interaction threshold $\vartheta \in \{0,\dots,F\}$ or the relative one ϑ/F. In [17,18] results are, for example, reported in terms of relative thresholds. The left inset of Fig. 3 confirms that the phase boundaries for different values of F as shown in the main panel do indeed show a reasonable collapse if plotted as a function of ϑ/F. At small values of ϑ systematic deviations are however observed. A different rescaling was suggested in [31], where results for the one-dimensional Axelrod model were shown to depend mostly on q/F. As demonstrated in the right inset of Fig. 3 an equally good collapse is observed in the $(\vartheta/F,q)$ plane, so that we here cannot reach a definitive conclusion as to whether there are any independent scaling parameters, and if so which ones they are [32]. Plotting the data in a plane spanned by ϑ and $\vartheta q/F$ (not shown here) indicates that $\vartheta q/F$ is roughly constant at the transition point.

Some further indications regarding the relevance of *absolute* as opposed to *relative* thresholds can be found in Fig. 4, where we show the density of fully saturated bonds P_F as a function of the density of initially active bonds $n_a(0) = \Sigma_{k=\vartheta+1}^{F-1} P_k(t=0)$. Simulations are here performed by fixing ϑ and F and by subsequently varying q. $n_a(0)$ then decreases
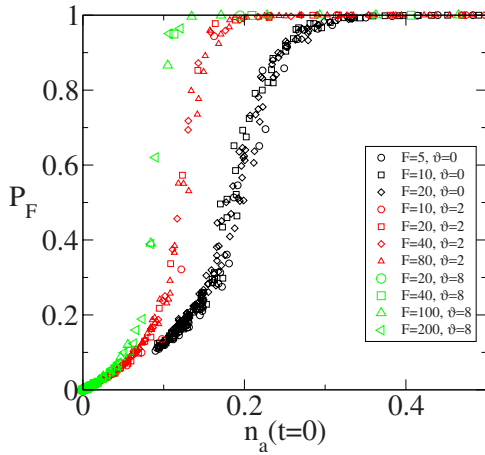
FIG. 4. (Color online) P_F versus fraction of active links $n_a(0)$ in initial configuration for nominal Axelrod dynamics at different ϑ and F. Data are from simulations (system size $N=20\times20$, run until convergence, and averages over ten samples are taken).

FIG. 5. (Color online) S/N versus time for $F=10, q=100$. $\vartheta =0,1,2$ in top, middle, and lower panels. Curves are shown for $r =10^{-2}$, 10^{-3}, and 10^{-4} (bottom to top at large times). From simulations, $N=10\times10$, and averages over ten runs. The x axes show time in multiples of N microscopic updates, i.e., after t every agent has on average been selected t times for potential update.

with increasing q. The data shown in the figure suggest a potential collapse on three different curves, one for each of the tested values $\vartheta=0,2,8$. While these findings might indicate some potential universality as F and q are varied at fixed ϑ, reaching a final conclusion as to whether absolute or relative thresholds are the relevant ones still remains an open question. It is also interesting to consider the case of large interaction thresholds, e.g., $\vartheta=F-2$, and to ask whether an ordered phase is found in finite systems. While we do not show results here, simulations at $F=10, \vartheta=8$ indicate that an extended ordered phase may here no longer be present. The case $q=1$ is trivially ordered, $q=2$ shows partial order, and no significant order is found at $q>2$. It is here important to notice that caution needs to be taken in the interpretation of results from the master equation, as the latter only applies to the mean-field system.

C. Effects of external noise

We now turn to a discussion of the effects of noise on nominal Axelrod dynamics. We will first focus on external noise as introduced above. This type of stochasticity describes random fluctuations which are triggered by an external event, and which result in individual spin components flipped randomly at a given rate. External noise was introduced in the context of the Axelrod model in [11], and the resulting random mutations describe what Axelrod refers to as "cultural drift" [1] in the population of agents. Klemm *et al.* [11] studied the effects of external noise as a function of the noise rate and system size for systems in stationary conditions. In *finite* systems a continuous transition between an ordered state at low noise rates r and a disordered state at large r has been identified in [11]. This transition relates to a characteristic relaxation time $T=\mathcal{O}(N\log N)$ in finite systems. When the noise rate is sufficiently large (larger than T^{-1}) stochastic perturbations build up in time and lead to a disordered state. For $rT\ll1$ the system drifts from one ordered state to another in time; time averaging effectively yields global order. Since at $q>q_c$ disorder is observed in the
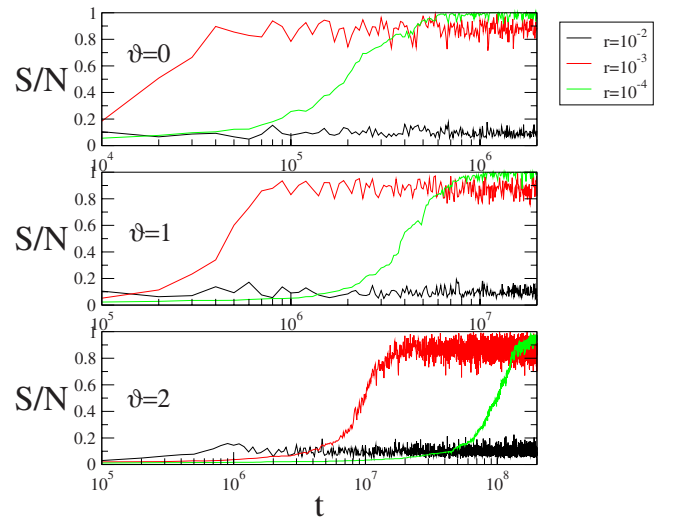
absence of noise ($r=0$), the behavior of the model in the limit $r\rightarrow0^+$ is discontinuous at $q>q_c$. The finiteness of the system is crucial here, so that the described behavior cannot be captured by the master equation.

The effects of external noise on finite Axelrod systems with general threshold are depicted in Fig. 5. The behavior of the model with confidence threshold is here found to be very similar to the one identified in [11] for conventional Axelrod dynamics. We here focus on $F=10, q=100$ as an example, but similar behavior can be expected for other model parameters in the disordered phase of the noise-free model. For small values of r the system orders after an initial transient. At large magnitude of the applied noise, no ordering is found, consistent with the results of [11]. This general qualitative picture appears to be independent of the applied threshold. The duration of the equilibration period in cases where the system orders, however, shows a significant dependence on the noise strength and on the chosen threshold. Generally, the time required to reach equilibration increases as r is lowered or as ϑ is increased (see Fig. 5). The value of S/N at equilibrium is a decreasing function of r. In the examples of Fig. 5 equilibration to a value of $S/N\approx0.1$ occurs fast at $r=10^{-2}$. At lower noise rates, S/N reaches values in the range of 0.8–1, but only after a substantial equilibration period, which increases as r is lowered. Only models with low or moderate threshold and/or sufficiently large noise strength can hence be found in stationary conditions in reasonable computing time. While analogy suggests that an ordered phase sets also at higher thresholds ϑ and small enough noise strengths if the dynamics is run long enough, we have not been able to confirm this explicitly due to computational limitations.

In the model at $\vartheta=0$, the transition in dependence on r just discussed disappears in infinite systems [11]. If, at $q >q_c$, the system size is taken to infinity *after* equilibration,
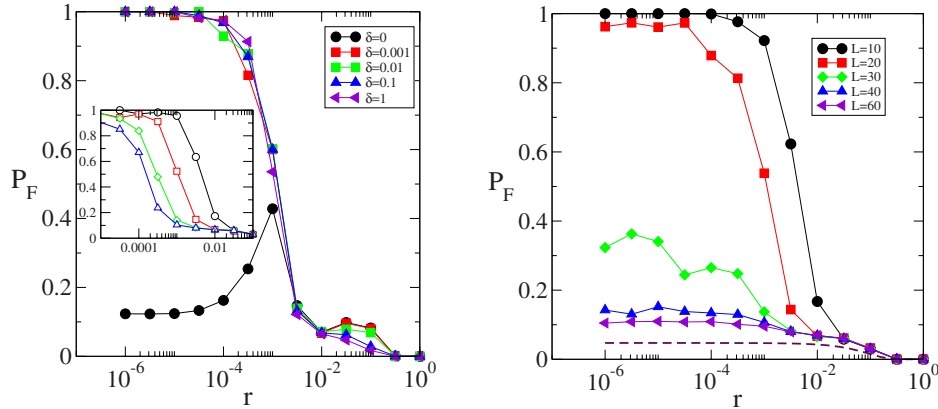
FIG. 6. (Color online) Effects of interaction noise. Left: P_F versus r for Axelrod model ($F=10$, $q=100$, $\vartheta=0$), $N=20\times20$, and run for 20 000 steps (averages over 10–20 samples). The inset shows results for systems of size $N=10\times10$, 20×20, 40×40, and 60×60 at $\delta=0.1$ from right to left. Right: P_F for different system sizes at fixed $\delta=0.1$ at $t=5000$. Dashed line in right panel is from numerical integration of the master equation.

i.e., if systems in a stationary state are considered at increasing N while keeping all other parameters fixed, the population always ends up in a disordered state for all $r\geq0$, as demonstrated in [11]. The discontinuity at $r\rightarrow0^+$ is removed. While we have not been able to obtain explicit verification of an analogous behavior in the model with nonzero interaction thresholds (due to the long involved equilibration times), simulations in models at nonzero interaction threshold combined with interaction noise (speeding up the dynamics) show that the transition induced by external noise is indeed a finite-size effect [see Sec. IV D and especially the inset of Fig. 6 (left panel)].

D. Effects of interaction noise

We next turn to a discussion of the effects of interaction noise, as parametrized by its amplitude δ. This type of stochasticity facilitates interaction, as it removes kinetic constraints and allows agents sharing ϑ or less opinions to interact (at rate δ), while in standard Axelrod dynamics they would not be able to align spin components. While the external noise of amplitude r has an ambiguous role of inducing order at low amplitudes and of driving the system to disorder at large r, interaction noise can generally be expected to favor order. It is hence interesting to study the system in the presence of both types of randomness and to identify reinforced ordering behavior or (at small r) potential competition between the ordering and disordering stochasticities (at large r).

Results for a nominal Axelrod system with both types of noise are reported in Fig. 6. The data indicate that the effects of interaction noise are mostly to facilitate order for large ranges of fixed external noise r. More specifically, the effects of interaction noise are to reduce the time scale on which finite systems order in the presence of external noise. The left panel of Fig. 6 shows the concentration P_F of bonds with full overlap in an Axelrod system run for a time which is not long enough for the system without interaction noise (circles, $\delta=0$) to develop order at the studied magnitudes r. Order at $r\lesssim0.001$ would develop only if the system were run for

longer times and had reached a stationary state. The curves for nonvanishing amplitude δ demonstrate the effect of interaction noise; the system now orders at small noise strengths r. While the qualitative behavior of the *stationary* system is not altered, the facilitation of the kinetic constraints drastically reduces equilibration times, and the system orders at sufficiently low r even after moderate running times. Interestingly, the noise strength r_0 separating the ordered from the disordered regime of the stationary system appears not to be affected much by the interaction noise. The curves displayed in the left panel of Fig. 6 are indeed mostly independent of δ, as long as $\delta>0$. It might potentially be interesting to study even lower δ, although probably unrealistic from the sociological point of view [33].

We conclude that the effect of interaction noise is to reduce relaxation times, but that it does not alter the phase behavior of the model, with an ordered phase at low r and a disordered one at large r. As in the absence of interaction noise ($\delta=0$) this transition is present only in finite systems, seen in the inset of the left panel of Fig. 6 and in the right panel. As the system size is increased at equilibrium the order-disorder crossover moves to smaller values of the noise strength r and can be expected to be absent in the thermodynamic limit, where only the disordered region prevails. Indeed rescaling of the data in the inset of the left panel demonstrates that $rN\ln N$ is the relevant scaling variable, similar to the observations of [11]. At finite running times t the order at low r is gradually reduced with increased system size and in the thermodynamic limit the system is qualitatively well described by the master equation.

Figure 7 finally confirms that this behavior is not limited to the standard Axelrod dynamics with vanishing interaction threshold ($\vartheta=0$). Interaction noise reduces the time scale on which the system orders at small r also in the model with moderate nonzero thresholds and that the system at $\vartheta>0$ behaves very much like the one at $\vartheta=0$.

V. EUCLIDEAN AXELROD DYNAMICS

We now turn to a modification of the Axelrod dynamics in which a metric allowing for gradual notions of agreement
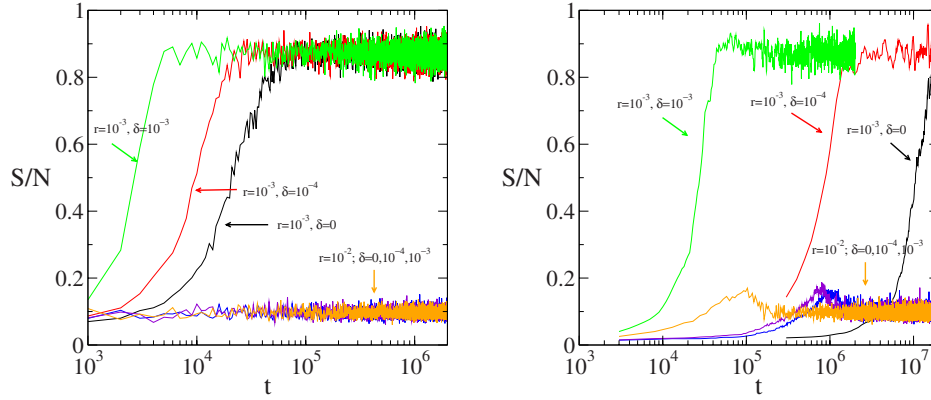
FIG. 7. (Color online) Effects of interaction noise. S/N versus time t at different levels of external and interaction noise. Data are from simulations of systems with $N=10\times10$ agents. The left part shows $\vartheta=0$ (averages over 50 samples) and the right one shows $\vartheta=2$ (10 runs and data smoothened by running averages). $F=10, q=100$ in both panels. At large r no order is reached, independent of δ. For small enough r the system orders; the time scale on which this occurs is largely reduced by the introduction of interaction noise and increases as the interaction threshold is raised (note the different scaling on the time axes on left and right).

between agents is introduced. As before, opinions take discrete values $\sigma_{if}\in\{1,\ldots,q\}$. However, the "distance" between two spins $\vec{\sigma}_i$ and $\vec{\sigma}_j$ is no longer measured in terms of the number of features on which the corresponding agents agree, but instead based on the following Euclidean distance between the spin vectors $\vec{\sigma}_i$ and $\vec{\sigma}_j$:

$$d = \sqrt{\frac{1}{F(q-1)^2}\sum_{f=1}^{F}(\sigma_{if}-\sigma_{jf})^2}. \quad (3)$$

Thus the distance between two agents ranges between 0 and 1. It takes the maximal value $d=1$ if and only if the opinions of the two agents are diametrically opposed, i.e., of for any feature f one has $\sigma_{if}=1$ and $\sigma_{jf}=q$ or $\sigma_{if}=q$ and $\sigma_{jf}=1$. Distances different from 0 or 1 thus indicate partial agreement between the two agents.

In the following we will take the potential of two neighboring agents to interact to be given by the following logit rule [34]:

$$p(d) = \frac{1}{1+e^{\beta(d-d_0)}}. \quad (4)$$

β is a control parameter allowing for the introduction of interaction noise. The case $\beta=\infty$ here corresponds to the noise-free (zero-temperature) case. If $\beta=\infty$, agents with distance $d>d_0$ are unable to interact, $p(d>d_0)=0$, whereas interaction always occurs for pairs of agents with distance $d<d_0$. d_0 is thus a threshold parameter, with large d_0 corresponding to a regime of strong confidence of agents in other people's opinions and small d_0 to cases in which interaction is rare. In order to avoid confusion let us at this point stress that the role of the threshold d_0 is inverse to the one of ϑ in the nominal Axelrod model: large ϑ makes interaction rare, whereas large d_0 facilitates spin updates.

Choosing finite values of β turns the hard threshold into a soft one. Interaction rates decrease smoothly with increasing distance. Crucially, at finite β, interaction is always possible, in principle, even if $d>d_0$. For $\beta=0$ finally, interaction is fully stochastic and independent of d. At any iteration, any

chosen pair of neighboring agents interacts with probability of 1/2.

Let us summarize the resulting dynamics as follows:

(i) Select one spin $i\in\{1,\ldots,N\}$ at random. Subsequently select one of its four nearest neighbors at random. Call this second spin j.

(ii) Compute the Euclidean distance $d(i,j)$ between i and j.

(iii) If $d(i,j)=0$ both agents agree on all features. Interaction has no effect. If $d\equiv d(i,j)>0$ then with probability $p(d)$ as defined above spins i and j interact as in the nominal Axelrod model: one feature f is chosen at random, so that $\sigma_{if}\neq\sigma_{jf}$. Then set $\sigma_{if}=\sigma_{jf}$.

(iv) External noise. With probability r perform the following: choose one spin i and one feature f at random. Set σ_{if} to a value chosen randomly from $\{1,\ldots,q\}$.

(v) Resume at (i).

A. Noise-free dynamics

The behavior of the noise-free system with Euclidean metric is described in Fig. 8. A transition between a disor-
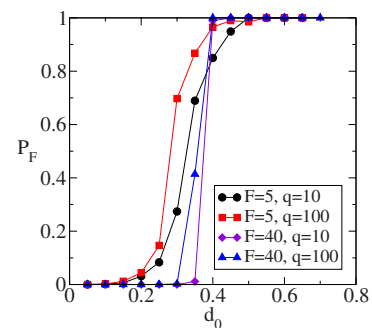

FIG. 8. (Color online) P_F versus interaction threshold d_0 for the Euclidean model. Data are from simulations at different values of F and q ($N=20\times20$ and averaged over ten samples). Simulations are run until each bond has distance $d=0$ or $d>d_0$, i.e., until no active bonds are left in the system.

dered phase at low thresholds d_0 and an ordered state at larger values of d_0 is observed. The behavior in these two phases is as follows: at low d_0 only neighboring agents with small differences in opinion can interact, so that the fraction of active bonds initially contained in the system is small. Dynamic arrest occurs quickly and the system remains disordered. At large enough thresholds d_0 the coarsening dynamics can persist until a fully ordered state is reached. Interestingly, as shown in Fig. 8, the critical value of the threshold d_{0c} does not depend much on the choice of F and q and takes values $d_{0c} \approx 0.4$. The transition appears to become sharper at larger values of F (see Fig. 8).

Plotting P_F versus q at fixed F suggests that q plays no significant role in the Euclidean model. Only for small values of q can a dependence of P_F on q be detected. This invariance is intuitively to be expected as d is normalized to range between 0 and 1 in the setup chosen here, so that q is merely a measure for how many discrete values can occur in between. Simulations with continuous opinions ranging in the interval $[0,1]$ (not shown here) reveal a behavior very similar to the one depicted in Fig. 8. We have also tested models with continuous opinions, in which both interacting agents agree on the *average* opinion of a given feature in case of interaction (with the same metric and kinetic constraints as before), and find similar behavior as a function of d_0. Similar models are discussed in [20,21], mostly focusing on the case of one feature.

B. Effects of external and interaction noise

The behavior of the Euclidean system under the influence of external and interaction noise is shown in Fig. 9. As seen in the main panel, a crossover between an ordered regime at low magnitudes r of the external noise and a disordered state at higher noise amplitudes is found, very much similar to the nominal Axelrod model. Interaction noise (finite β) appears to have only little effect on this crossover for all values tested. Due to long equilibration times we have not performed a full analysis of the impact of external noise ($r > 0$) in the large-N limit of the model zero temperature ($\beta = \infty$). The inset of Fig. 9 however demonstrates that at finite β the range of r in which the system orders is reduced as the system size is increased, similar to what is found in the nominal Axelrod model. The ordering behavior at small values of the external noise strength hence again appears to be present only in finite systems.

VI. SUMMARY AND CONCLUSIONS

We have extended Axelrod model for social influence to include varying interaction threshold, noise, and metric features. In the basic model, which is two-dimensional, individuals are represented as multicomponent spins and interact to become culturally closer, starting from a disordered random initial state. In our extension external and interaction noises are introduced, a confidence threshold limits the interaction, and a notion of distance between opinions is considered. We find that the confidence threshold does not influence the qualitative behavior of the model much and that the
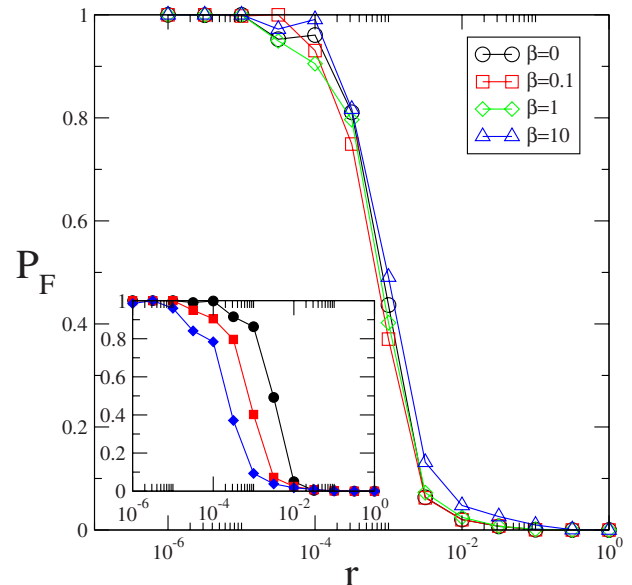

FIG. 9. (Color online) Effects of noise on the Euclidean model. Main panel shows P_F versus r from simulations of a system of size $N = 20 \times 20$, run for 200 000 macroscopic steps, and averages over ten samples ($F = 10, q = 100$). The interaction threshold is $d_0 = 0.3$, i.e., the system is in a mostly disordered phase if $\beta = \infty$ ($P_F \approx 0.2$). Inset: P_F versus r at fixed $\beta = 1$ for system sizes $N = 40^2$, 20^2, and 10^2 from left to right (same simulation parameters as in main panel and five samples only for $N = 40^2$).

typical transition, separating an ordered and a disordered equilibrium state, is preserved and determined basically by the initial probability of interaction between two individuals. The threshold limits the probability of interaction and hence favors disorder. The introduction of an external noise brings in finite systems a continuous transition between an order-favoring and a disorder-favoring role of the noise, according to whether the noise is small or large, respectively, independent of the threshold. This implies a discontinuity, in the region where the final state would be disordered in the absence of external noise. Such a discontinuity is removed in the thermodynamic limit as we have verified in models with interaction noise. The latter type of noise favors order by reducing relaxation times and does not alter the phase structure of the model. The other variant we study is the one with a notion of distance between opinions. We find that the model exhibits an order-disorder transition as the distance threshold is varied, consistent with the idea that the transition is the result of the competition between an ordering dynamics (the relative importance of which is determined by the distance threshold) and an initial disorder (as measured by size of the space from which the starting configuration is drawn at random).

Although we do not report the results here, the introduction of heterogeneous confidence thresholds in the context of metric Axelrod systems allows one to study e.g., the question of whether extremism can prevail in such models. We found that the presence of a small fraction of individuals with a sufficiently rooted opinion can drive the whole population to the extreme ends of the opinion spectrum. Further applications of heterogeneous interaction thresholds and metric fea-

tures might include extensions addressing immigration or geographic barriers. Immigrants can, for example, be assumed to be more likely to interact with other immigrants than with members of the original population, and geographical barriers can be modeled by suppressing interaction at certain locations in space. This would lead to different interaction thresholds and tolerance levels, modulated either in space or dependent on the two agents picked for potential interaction.

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[29] We here do not impose the constraint that R be connected as it is often done in the literature. This simplifies the numerical determination of the size of the largest homogeneous region. Results regarding the phase structure do not depend on whether or not connectedness is assumed in the definition of homogeneous regions. In particular the ordered phase consists of one region spanning the entire system, so that both definitions yield the same characterization of the low-q phase.

[30] Integration of the master equation yields a positive value for the fraction of active bonds in the ordered phase (coarsening lasts indefinitely in infinite systems in the ordered phase) and a vanishing fraction in the disordered phase, similar to [8]. The location of the phase transition in Fig. 3 is obtained as the point where the number of active bonds decreases below a threshold on the order of 10^{-3}, or equivalently as the point at which a discontinuity in P_F or its derivative occurs.

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[32] Rescaling both axes in Fig. 3 and plotting the phase boundaries in $(\vartheta/\sqrt{F}, q/\sqrt{F})$ plane give an equally good collapse.

[33] We here note that the data shown in Fig. 6 suggest that P_F at stationarity might potentially be a decreasing function of δ for $r \gtrsim 0.01$. P_F is small for all tested values of δ in this regime, however, so that we have not investigated this regime further.

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